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The τ -function of the quadrilateral lattice

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Abstract

We investigate the τ -function of the quadrilateral lattice using the nonlocal $\bar{\partial}$ -dressing method, and we show that it can be identified with the Fredholm determinant of the integral equation which naturally appears within that approach.

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1. Introduction

Lattices of planar quadrilaterals were introduced into integrability theory in [9, 13] as discrete analogs [28, 29] of conjugate nets [7]. The basic nonlinear system describing such lattices, the discrete Darboux equations (1.2) or (1.6), however first appeared in [4] as generic discrete systems integrable by the nonlocal $\bar{\partial}$ -dressing method [1, 3, 22, 33]. The classical Darboux equations [7], nowadays also called the N -wave system [20, 33], form the basic system of equations of the multicomponent Kadomtsev–Petviashvili (KP) hierarchy [8, 20]. The Darboux equations were rediscovered [33] in a generalized matrix form, as the basic set of equations solvable by the $\bar{\partial}$ -dressing method. For example, the KP equation [8] was shown in [3] to be a limiting case of the Darboux system. In fact the whole KP hierarchy can be written, in the so-called Miwa coordinates, as an infinite system of the Darboux equations [5].

In [11] it was shown that the τ -function of conjugate nets, which is a potential apparently already known to Darboux, and which can be viewed as the τ -function of the whole multicomponent KP hierarchy upon identification of the higher times of the hierarchy with isoconjugate deformations of the nets [17], can be identified with the Fredholm determinant of the integral equation inverting the nonlocal $\bar{\partial}$ -problem as applied to Darboux equations. The τ -functions play the central role [2, 6, 8, 18, 21, 25, 26, 30, 31] in establishing the connections between integrable systems and quantum field theory, statistical mechanics or the theory of random matrices. They are often represented as determinants of infinite matrices or can be identified with the Fredholm determinant of the integral Gel'fand–Levitan–Marchenko

equation used to solve the model under consideration. Within the context of the Zakharov and Shabat dressing method [35], the τ -function of the KP hierarchy was interpreted as the Fredholm determinant in [27].

The $\bar{\partial}$ -dressing and related methods have been applied to conjugate nets, quadrilateral lattices and their reductions and transformations in a number of papers, including apart from the above mentioned also [14–17, 24, 34]. The main result of the present paper, somehow missed in earlier studies, is the interpretation of the τ -function of the quadrilateral lattice as the Fredholm determinant of the integral equation inverting the nonlocal $\bar{\partial}$ -problem as studied in [4]. In fact, as the discrete Darboux equations can be obtained from the multicomponent KP hierarchy via the Miwa transformation [17], the result is not a surprise. However, because integrable discrete systems are often considered as more fundamental than their differential counterpart, and because the nonlocal $\bar{\partial}$ -approach contains, as particular reductions, earlier versions of the inverse spectral (scattering) method, it was desirable to have a direct proof.

The paper is constructed as follows. In section 2, we collect the basic elements of the $\bar{\partial}$ -dressing method and we recall the way of solving the discrete Darboux equations within this method. Section 3 is devoted to a presentation of the Fredholm determinant interpretation of the τ -function of the quadrilateral lattice. In the remaining part of section 1, we present the basic elements of the quadrilateral lattice theory.

In affine representation the quadrilateral lattice is a mapping $\mathbf{x} : \mathbb{Z}^N \rightarrow \mathbb{R}^M$, and the planarity of its elementary quadrilaterals can be formulated in terms of the system of the discrete Laplace equations:

$$\Delta_i \Delta_j \mathbf{x} = (T_i A_{ij}) \Delta_i \mathbf{x} + (T_j A_{ji}) \Delta_j \mathbf{x}, \quad i \neq j, \quad i, j = 1, \dots, N, \quad (1.1)$$

where T_i denotes the shift operator in n_i and $\Delta_i = T_i - 1$ is the corresponding partial difference. For $N > 2$, the coefficients A_{ij} of the Laplace equation (1.1) satisfy the discrete Darboux equations:

$$\Delta_k A_{ij} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik} - (T_k A_{ij}) A_{ik}, \quad i \neq j \neq k \neq i, \quad (1.2)$$

which is the integrable discrete analog of the Darboux equations describing multidimensional conjugate nets [7]. Equations (1.2) imply the existence of the potentials H_i [4, 13]:

$$A_{ij} = \frac{\Delta_j H_i}{H_i}, \quad i \neq j, \quad (1.3)$$

which, in analogy to the continuous case, are called the Lamé coefficients.

Introduce the suitably scaled tangent vectors $\mathbf{X}_i, i = 1, \dots, N$,

$$\Delta_i \mathbf{x} = (T_i H_i) \mathbf{X}_i, \quad (1.4)$$

then

$$\Delta_j \mathbf{X}_i = (T_j Q_{ij}) \mathbf{X}_j, \quad i \neq j, \quad (1.5)$$

and the compatibility condition for the system (1.5) gives the following new form of the discrete Darboux equations:

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i \neq j \neq k \neq i. \quad (1.6)$$

The Lamé coefficients H_i solve the linear equations

$$\Delta_i H_j = (T_i H_i) Q_{ij}, \quad i \neq j, \quad (1.7)$$

whose compatibility again gives equations (1.6).

Define [14, 23] the potentials ρ_i as solutions to compatible equations:

$$\frac{T_j \rho_i}{\rho_i} = 1 - (T_i Q_{ji})(T_j Q_{ij}), \quad i \neq j. \quad (1.8)$$

The right-hand side of equation (1.8) is symmetric with respect to the interchange of i and j , which implies the existence of a potential τ , such that

$$\rho_i = \frac{T_i \tau}{\tau}, \tag{1.9}$$

which is called the τ -function of the quadrilateral lattice [14].

2. The $\bar{\partial}$ -dressing method and the discrete Darboux equations

In this section, we recall [3, 22, 33] the basic idea of the nonlocal $\bar{\partial}$ -dressing method in application to the quadrilateral lattice and the discrete Darboux equations [4].

Consider the following integro-differential equation in the complex plane \mathbb{C} :

$$\bar{\partial}\chi(\lambda) = \bar{\partial}\eta(\lambda) + \int_{\mathbb{C}} R(\lambda, \lambda')\chi(\lambda') d\lambda' \wedge d\bar{\lambda}', \tag{2.1}$$

where $R(\lambda, \lambda')$ is a given $\bar{\partial}$ datum, which decreases quickly enough at ∞ in λ and λ' , and the function $\eta(\lambda)$, the normalization of the unknown $\chi(\lambda)$, is a given rational function, which describes the polar behavior of $\chi(\lambda)$ in \mathbb{C} and its behavior at ∞ :

$$\chi(\lambda) - \eta(\lambda) \rightarrow 0, \quad \text{for } |\lambda| \rightarrow \infty.$$

We remark that the dependence of $\chi(\lambda)$ and $R(\lambda, \lambda')$ on $\bar{\lambda}$ and $\bar{\lambda}'$ will be systematically omitted, for notational convenience.

Due to the generalized Cauchy formula, the nonlocal $\bar{\partial}$ problem (2.1) is equivalent to the following Fredholm integral equation of the second kind:

$$\chi(\lambda) = \eta(\lambda) - \int_{\mathbb{C}} K(\lambda, \lambda')\chi(\lambda') d\lambda' \wedge d\bar{\lambda}', \tag{2.2}$$

with the kernel

$$K(\lambda, \lambda') = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{R(\lambda'', \lambda')}{\lambda'' - \lambda} d\lambda'' \wedge d\bar{\lambda}'' . \tag{2.3}$$

Recall (see, for example, [32]) that the Fredholm determinant D_F is defined by the series

$$D_F = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{C}^m} K \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_m \\ \zeta_1 & \zeta_2 & \cdots & \zeta_m \end{pmatrix} d\zeta_1 \wedge d\bar{\zeta}_1 \cdots d\zeta_m \wedge d\bar{\zeta}_m, \tag{2.4}$$

where

$$K \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{pmatrix} = \det(K(\zeta_i, \mu_j))_{1 \leq i, j \leq m}.$$

For a nonvanishing Fredholm determinant, the solution of (2.2) can be written in the form

$$\chi(\lambda) = \eta(\lambda) - \int_{\mathbb{C}} \frac{D_F(\lambda, \lambda')}{D_F} \eta(\lambda') d\lambda' \wedge d\bar{\lambda}', \tag{2.5}$$

where the Fredholm minor is defined by the series

$$D_F(\lambda, \lambda') = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{C}^m} K \begin{pmatrix} \lambda & \zeta_1 & \cdots & \zeta_m \\ \lambda' & \zeta_1 & \cdots & \zeta_m \end{pmatrix} d\zeta_1 \wedge d\bar{\zeta}_1 \cdots d\zeta_m \wedge d\bar{\zeta}_m. \tag{2.6}$$

Let $\lambda_i^{\pm} \in \mathbb{C}$, $i = 1, \dots, N$, be pairs of distinct points of the complex plane. To get the $\bar{\partial}$ -dressing method of construction of solutions to the discrete Darboux equations, one introduces [4] the following dependence of the kernel R on the variables $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$:

$$T_i R(\lambda, \lambda'; n) = L_i(\lambda)^{-1} R(\lambda, \lambda'; n) L_i(\lambda'), \quad L_i(\lambda) = \frac{\lambda - \lambda_i^-}{\lambda - \lambda_i^+}, \tag{2.7}$$

or equivalently

$$R(\lambda, \lambda'; n) = G(\lambda; n)^{-1} R_0(\lambda, \lambda') G(\lambda'; n), \quad G(\lambda; n) = \prod_{i=1}^N L_i(\lambda)^{n_i}, \quad (2.8)$$

where $R_0(\lambda, \lambda')$ is independent of n . We assume that R_0 decreases at λ_i^\pm and in poles of the normalization function η fast enough such that $\chi - \eta$ is regular at these points [3, 4].

Remark. In the paper, we always assume that the kernel R in the nonlocal $\bar{\partial}$ problem is such that the Fredholm equation (2.2) is uniquely solvable. Then, by the Fredholm alternative, the homogenous equation with $\eta = 0$ has only a trivial solution.

Directly one can verify the following result which gives evolution of the kernel K of the Fredholm equation (2.2) implied by evolution of the $\bar{\partial}$ datum.

Lemma 1. *The evolution (2.7) of the kernel R implies that the kernel K of the integral equation (2.2) is subject to the equation*

$$T_i K(\lambda, \lambda'; n) = L_i(\lambda)^{-1} [K(\lambda, \lambda'; n) + (L_i(\lambda) - 1) K(\lambda_i^-, \lambda'; n)] L_i(\lambda'); \quad (2.9)$$

moreover,

$$T_i K(\lambda_i^+, \lambda'; n) = K(\lambda_i^-, \lambda'; n) L_i(\lambda'). \quad (2.10)$$

This leads to the following crucial, for our purposes, result.

Lemma 2. *When $\chi(\lambda; n)$ is the unique, by assumption, solution to the $\bar{\partial}$ problem (2.1) with the kernel R evolving according to (2.7), and with normalization $\eta(\lambda; n)$, then the function $L_i(\lambda) T_i \chi(\lambda; n)$ is the solution to the same $\bar{\partial}$ problem but with the new normalization*

$$\eta^{(i)}(\lambda; n) = L_i(\lambda) T_i \eta(\lambda; n) + (L_i(\lambda) - 1) \lim_{\lambda \rightarrow \lambda_i^+} (T_i \chi(\lambda; n) - T_i \eta(\lambda; n)). \quad (2.11)$$

Remark. We allow for the normalization to have poles at the distinguished points λ_i^\pm of the construction.

Proof. Application of equations (2.9) and (2.10) to the shifted formula (2.2) leads to

$$\begin{aligned} L_i(\lambda) T_i \chi(\lambda; n) &= L_i(\lambda) T_i \eta(\lambda; n) - \int_{\mathbb{C}} K(\lambda, \lambda'; n) L_i(\lambda') T_i \chi(\lambda'; n) d\lambda' \wedge d\bar{\lambda}' \\ &\quad - (L_i(\lambda) - 1) \int_{\mathbb{C}} T_i K(\lambda_i^+, \lambda'; n) L_i(\lambda') T_i \chi(\lambda'; n) d\lambda' \wedge d\bar{\lambda}'. \end{aligned}$$

To obtain the statement, note that

$$\lim_{\lambda \rightarrow \lambda_i^+} (T_i \chi(\lambda; n) - T_i \eta(\lambda; n)) = - \int_{\mathbb{C}} T_i K(\lambda_i^+, \lambda'; n) L_i(\lambda') T_i \chi(\lambda'; n) d\lambda' \wedge d\bar{\lambda}'. \quad \square$$

The following result allows us to give the $\bar{\partial}$ -method of construction of solutions to the discrete Darboux equations

Proposition 3. *Let $\chi_i(\lambda; n), i = 1, \dots, N$, be a solution to the $\bar{\partial}$ problem (2.1) with the normalization*

$$\eta_i(\lambda) = L_i(\lambda) - 1 = \frac{\lambda_i^+ - \lambda_i^-}{\lambda - \lambda_i^+}. \quad (2.12)$$

Denote

$$G_i(\lambda; n) = \prod_{j=1, j \neq i}^N L_j(\lambda)^{n_j}; \tag{2.13}$$

then the functions

$$\psi_i(\lambda; n) = \chi_i(\lambda; n)G(\lambda; n)G_i(\lambda_i^+; n)^{-1}, \tag{2.14}$$

$$Q_{ij}(n) = \chi_i(\lambda_j^+; n) \frac{G_j(\lambda_j^+; n)}{G_i(\lambda_i^+; n)} L_j(\lambda_i^+) \tag{2.15}$$

satisfy equations

$$\Delta_j \psi_i(\lambda; n) = T_j Q_{ij}(n) \psi_j(\lambda; n), \quad j \neq i, \tag{2.16}$$

$$\Delta_j Q_{ik}(n) = T_j Q_{ij}(n) Q_{jk}(n), \quad i \neq j \neq k \neq i. \tag{2.17}$$

Proof. The combination $L_j(\lambda)T_j\chi_i(\lambda; n)L_j(\lambda_i^+)^{-1} - \chi_i(\lambda; n)$, $j \neq i$, satisfies the integral equation (2.2) with the same normalization as $T_j\chi_i(\lambda_i^+)\chi_j(\lambda; n)$. Therefore, by the Fredholm alternative,

$$L_j(\lambda)T_j\chi_i(\lambda; n)L_j(\lambda_i^+)^{-1} - \chi_i(\lambda; n) = T_j\chi_i(\lambda_i^+)\chi_j(\lambda; n), \quad j \neq i, \tag{2.18}$$

which leads to the linear system (2.16). Its compatibility (2.17) can also be obtained by evaluating equation (2.18) at the points λ_k^+ , $k \neq i, j$. \square

For completeness, we also recall the following result of [4].

Theorem 4. Let $\chi(\lambda; n)$ be a solution to the $\bar{\partial}$ problem (2.1) with the canonical normalization $\eta(\lambda) = 1$; then the function

$$\psi(\lambda; n) = \chi(\lambda; n)G(\lambda; n)$$

satisfies the discrete Laplace system

$$\Delta_i \Delta_j \psi(\lambda; n) = (T_i A_{ij})(n) \Delta_i \psi(\lambda; n) + (T_j A_{ji})(n) \Delta_j \psi(\lambda; n), \quad i \neq j,$$

with coefficients

$$A_{ij}(n) = L_j(\lambda_i^+) \frac{T_j \chi(\lambda_i^+; n)}{\chi(\lambda_i^+; n)} - 1, \quad i \neq j,$$

while the corresponding Lamé coefficients are given by

$$H_i(n) = \chi(\lambda_i^+; n)G_i(\lambda_i^+; n).$$

Remark. Various n -independent measures $d\mu_a$ on \mathbb{C} give rise to coordinates

$$x^a(n) = \int_{\mathbb{C}} \psi(\lambda; n) d\mu_a,$$

of quadrilateral lattices, having $H_i(n)$ as the Lamé coefficients, and the functions

$$X_i^a(n) = \int_{\mathbb{C}} \psi_i(\lambda; n) d\mu_a$$

being coordinates of the normalized tangent vectors. To obtain real lattices, the kernel R_0 , the points λ_j^\pm and the measures $d\mu_a$ should satisfy certain additional conditions.

3. The first potentials and the τ -function

To give the meaning of the τ -function within the $\bar{\partial}$ -dressing method, we first present the meaning of the potentials ρ_i defined by equations (1.8).

Proposition 5. *Within the $\bar{\partial}$ -dressing method, the potentials ρ_i can be identified with*

$$\rho_i(n) = -\chi_i(\lambda_i^-; n) \frac{G_i(\lambda_i^-; n)}{G_i(\lambda_i^+; n)}. \tag{3.1}$$

Proof. Evaluation of formula (2.18) at $\lambda = \lambda_j^-$ gives

$$\chi_i(\lambda_j^-; n) = -T_j \chi_i(\lambda_j^+; n) \chi_j(\lambda_j^-; n). \tag{3.2}$$

Evaluation in turn of (2.18) at $\lambda = \lambda_i^-$ gives

$$T_j \chi_i(\lambda_i^-; n) \frac{L_j(\lambda_i^-)}{L_j(\lambda_i^+)} - \chi_i(\lambda_i^-; n) = T_j \chi_i(\lambda_j^+; n) \chi_j(\lambda_i^-; n),$$

which in view of (3.2) implies

$$T_j \chi_i(\lambda_i^-; n) \frac{L_j(\lambda_i^-)}{L_j(\lambda_i^+)} = \chi_i(\lambda_i^-; n) (1 - T_j \chi_i(\lambda_j^+; n) T_i \chi_j(\lambda_i^+; n)).$$

The last formula, the identification (2.15) and equation (1.8) give the statement. □

Before proving that the τ -function of the quadrilateral lattice is equal essentially to the Fredholm determinant of the integral equation inverting the nonlocal $\bar{\partial}$ problem (2.1) with the $\bar{\partial}$ datum evolving according to the rule (2.7), we will need the following technical result.

Lemma 6. *The evolution (2.9) of the kernel of the Fredholm equation implies the following evolution of the determinants in the series defining the Fredholm determinant D_F :*

$$\begin{aligned} T_i K \left(\begin{array}{cccc|c} \zeta_1 & \zeta_2 & \cdots & \zeta_m & n \\ \zeta_1 & \zeta_2 & \cdots & \zeta_m & \end{array} \right) &= K \left(\begin{array}{cccc|c} \zeta_1 & \zeta_2 & \cdots & \zeta_m & n \\ \zeta_1 & \zeta_2 & \cdots & \zeta_m & \end{array} \right) \\ &+ \sum_{j=1}^m [L_i(\zeta_j) - 1] K \left(\begin{array}{cccc|c} \lambda_i^- & \zeta_1 & \cdots & \check{\zeta}_j & \cdots & \zeta_m & n \\ \zeta_j & \zeta_1 & \cdots & \check{\zeta}_j & \cdots & \zeta_m & \end{array} \right), \end{aligned}$$

where the symbol $\check{\zeta}_j$ means that ζ_j should be removed from the sequence.

Proof. Extracting $L_i(\zeta_j)^{-1}$ from rows and $L_i(\zeta_k)$ from columns of the determinant

$$\begin{aligned} T_i K \left(\begin{array}{cccc|c} \zeta_1 & \cdots & \zeta_m & & n \\ \zeta_1 & \cdots & \zeta_m & & \end{array} \right) \\ = \det \left(L_i(\zeta_j)^{-1} [K(\zeta_j, \zeta_k; n) + (L_i(\zeta_j) - 1)K(\lambda_i^-, \zeta_k; n)] L_i(\zeta_k) \right)_{1 \leq j, k \leq m}, \end{aligned}$$

one arrives at the bordered determinant (an object often encountered in soliton theory [19])

$$\begin{aligned} \det \left(K(\zeta_j, \zeta_k; n) + (L_i(\zeta_j) - 1)K(\lambda_i^-, \zeta_k; n) \right)_{1 \leq j, k \leq m} \\ = \det(K(\zeta_j, \zeta_k; n))_{1 \leq j, k \leq m} + \sum_{j, k=1}^m (L_i(\zeta_j) - 1) \Delta_{jk} K(\lambda_i^-, \zeta_k; n), \end{aligned}$$

where by $(\Delta_{jk})_{1 \leq j, k \leq m}$ we denote the cofactor matrix of $(K(\zeta_j, \zeta_k; n))_{1 \leq j, k \leq m}$. Note that

$$\sum_{k=1}^m \Delta_{jk} K(\lambda_i^-, \zeta_k; n) = K \left(\begin{array}{cccc|c} \zeta_1 & \cdots & \lambda_i^- & \cdots & \zeta_m \\ \zeta_1 & \cdots & \zeta_j & \cdots & \zeta_m \end{array} \middle| n \right),$$

and, finally, application of an even number of transpositions to the above determinant concludes the proof. \square

From lemma 6, we immediately obtain the evolution rule of the Fredholm determinant.

Proposition 7. *The evolution (2.9) of the kernel of the Fredholm equation implies the following evolution of the Fredholm determinant D_F :*

$$T_i D_F(n) = D_F(n) + \int_{\mathbb{C}} D_F(\lambda_i^-, \lambda; n) [L_i(\lambda) - 1] d\lambda \wedge d\bar{\lambda}. \tag{3.3}$$

We are ready to state the main result of the paper.

Theorem 8. *Within the $\bar{\delta}$ -dressing method with the $\bar{\delta}$ datum evolving according to the rule (2.7) the τ -function of the quadrilateral lattice can be identified, up to a standard factor, with the Fredholm determinant as follows:*

$$\tau(n) = D_F(n) \prod_{i < j} C_{ij}^{n_i n_j}, \quad \text{where} \quad C_{ij} = \frac{L_i(\lambda_j^-)}{L_i(\lambda_j^+)} = C_{ji}. \tag{3.4}$$

Proof. Proposition 7, formula (2.5) and proposition 5 imply that

$$\frac{T_i D_F(n)}{D_F(n)} = -\chi_i(\lambda_i^-; n) = \rho_i(n) \frac{G_i(\lambda_i^+; n)}{G_i(\lambda_i^-; n)}.$$

Due to equation (1.9), we have

$$\frac{T_i \tau(n)}{\tau(n)} = \frac{T_i D_F(n)}{D_F(n)} \prod_{j=1, j \neq i}^N \left(\frac{L_i(\lambda_j^-)}{L_i(\lambda_j^+)} \right)^{n_j},$$

which upon integration gives the statement of the theorem. \square

4. Conclusion and remarks

We have shown that within the $\bar{\delta}$ -dressing method, the τ -function of the quadrilateral lattice can be identified with the Fredholm determinant of the integral equation inverting the corresponding nonlocal $\bar{\delta}$ problem. The proof of this fact was quite elementary, and in a certain aspect (in the continuous limit λ_i^- and λ_i^+ coincide, which produces essential singularities in the function $G(\lambda)$) even simpler than in an earlier similar paper [11] where the τ -function of the conjugate nets had been treated.

It should be noted that the prescribed singularity structure of the functions $\psi_i(\lambda; n)$ is the same as in the construction of the algebro-geometric solutions of the discrete Darboux equations [10], and the $\bar{\delta}$ -dressing method meaning of the first potentials ρ_i given here is a direct counterpart of that given in [10].

Finally, we remark that geometrically one can consider quadrilateral lattices in projective spaces over division rings [12] and a substantial part of the integrability structures (including the existence of the first potentials ρ_i or the Darboux-type transformations) goes over. However,

it seems that without certain additional (commutativity) conditions one cannot define a τ -function of such a quadrilateral lattice over a general division ring.

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